THE GLOBAL INDICES OF LOG CALABI-YAU VARIETIES –A SUPPLEMENT TO FUJINO'S PAPER: THE INDICES OF LOG CANONICAL SINGULARITIES–

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Abstract. This paper gives the all possible global indices of log Calabi-Yau 3-folds with standard coefficients on the boundaries and having lc, non-klt singularities. This follows easily from the discussion in the paper: The indices of log canonical singularities by Fujino.

1. Introduction

In this paper, we study a log pair (X, B_X) with a normal projective variety X defined over $\mathbb{C}$ and a boundary B_X of standard coefficients (i.e., $B_X = \sum b_i B_i$, where $b_i = 1$ or $1 - 1/m$ for $m \in \mathbb{N}$). A pair (X, B_X) is called a log Calabi-Yau variety if it has lc singularities and $K_X + B_X \equiv 0$. For a log Calabi-Yau variety (X, B_X) assume that there exists $r \in \mathbb{N}$ such that $r(K_X + B_X) \sim 0$. (For $\dim X \leq 3$ this holds true for every log Calabi-Yau variety, by the abundance theorem ([6, 11.1.3] and [5]). We define the global index $\mathrm{Ind}(X, B_X)$ by the minimum of such r.

It is well known that a non-singular surface X with $K_X \equiv 0$ has $\mathrm{Ind}(X, 0) = 1, 2, 3, 4, 6$. Blache [2] proved that a normal surface X with $K_X \equiv 0$ and having lc non-klt singularity has also $\mathrm{Ind}(X, 0) = 1, 2, 3, 4, 6$. This is generalized into the case that log Calabi-Yau surface (X, B_X) has lc and non-klt singularities in [12, 2.3].

In this paper we prove the following:

Theorem 1.1. *Let (X, B_X) be a log Calabi-Yau 3-fold with lc non-klt singularities. Then $r \in \mathbb{N}$ can be the global index $\mathrm{Ind}(X, B_X)$, if and only if $\varphi(r) \leq 20$ and $r \neq 60$, where φ is the Euler function. In particular the global index is bounded.*

This theorem is a corollary of the following:

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Theorem 1.2. *Assume the Abundance Theorem and G-equivariant log Minimal Model Program for dimension $\leq n$, where G is a finite group. Let (X, B_X) be an n-dimensional log Calabi-Yau variety with non-klt singularities. If the conjectures (F'_j) and (F_l) in* [3] *hold true for $j = n-1$, $l \leq n-2$, then the global index $\mathrm{Ind}(X, B_X)$ is bounded.*

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2. The global indices

2.1. Throughout this paper, we use the notation and the terminologies in [3]. We assume the Abundance Theorem and the G-equivariant log Minimal Model Program (as is well known, these hold for dimension ≤ 3 by [6, 11.1.3], [5] and [7, 2.21]).

2.2. Let (X, B_X) be an n-dimensional log Calabi-Yau variety. Since we assume the Abundance Theorem, there exists $r \in \mathbb{N}$ such that $r(K_X + B_X) \sim 0$. Let $\pi : (Y, B) \to (X, B_X)$ be the index 1 cover with

$$K_Y + B = \pi^*(K_X + B_X).$$

Here the index 1 cover is constructed as follows: let $r = \mathrm{Ind}(X, B_X)$, then there exists a rational function φ on X such that $r(K_X + B_X) = \mathrm{div}(\varphi)$; take the integral closure Y in $K(X)(\sqrt[r]{\varphi})$. Note that $K_Y + B \sim 0$, that $B = \pi^*(\lfloor B_X \rfloor)$ is a reduced divisor and that π ramifies only over the components of B_X whose coefficients are < 1, as the coefficients of B_X are standard. Since $K_X + B_X$ is lc (resp. klt) if and only if $K_Y + B$ is lc (resp. klt), (Y, B) is log Calabi-Yau of global index 1. Therefore we obtain that every log Calabi-Yau variety (X, B_X) is the quotient of a log Calabi-Yau variety of global index 1 by the action of a finite cyclic group.

2.3. Let G be the cyclic group acting on a log Calabi-Yau variety (Y, B) of global index 1. Since G acts on $\Gamma(Y, K_Y + B) = \mathbb{C}$, there is a corresponding representation $\rho : G \to GL(\Gamma(Y, K_Y + B)) = \mathbb{C}^*$.

Lemma 2.4. *Under the notation above, Let (X, B_X) be the quotient $(Y, B)/G$ by G. Then*

$$\mathrm{Ind}(X, B_X) = |\mathrm{Im}\rho|.$$

Proof. For a generator $\theta \in \Gamma(Y, K_Y + B)$, $\theta^{|\mathrm{Im}\rho|}$ is G-invariant, therefore $\Gamma(X, |\mathrm{Im}\rho|(K_X + B_X)) \neq 0$, which yields $\mathrm{Ind}(X, B_X) \leq |\mathrm{Im}\rho|$. Conversely, for a generator $\eta \in \Gamma(X, \mathrm{Ind}(X, B_X)(K_X + B_X))$, $\pi^*\eta \in \Gamma(Y, \mathrm{Ind}(X, B_X)(K_Y + B))$ is G-invariant. If we write $\pi^*\eta = a\theta^{\mathrm{Ind}(X,B_X)}$ ($a \in \mathbb{C}$), for a generator $g \in G$, $(a\theta^{\mathrm{Ind}(X,B_X)})^g = a\epsilon^{\mathrm{Ind}(X,B_X)}\theta^{\mathrm{Ind}(X,B_X)} = a\theta^{\mathrm{Ind}(X,B_X)}$, where ϵ is a primitive $|\mathrm{Im}\rho|$-th root of unity. Hence, $\mathrm{Ind}(X, B_X) \geq |\mathrm{Im}\rho|$. □

2.5. Now we are going to study lc and non-klt log Calabi-Yau varieties. Let (Y, B) be an n-dimensional log Calabi-Yau variety of global index 1 with lc and non-klt singularities. Assume that a cyclic group G acts on (Y, B). Then we have a projective G-equivariant log resolution $\varphi : \tilde{Y} \to Y$ of (Y, B). Indeed, let $\varphi' : \tilde{Y}' \to Y$ be the canonical resolution of (Y, B) constructed in [1], then φ' is projective and ${\varphi'}^{-1}(B)\cup$(the exceptional set) is normal crossing divisor. By the blow up at a suitable G-invariant center, we obtain the divisor with simple normal crossings. Define the subboundary F on $\tilde{Y}$ by $K_{\tilde{Y}} + F = \varphi^*(K_Y + B)$. Run G-equivariant log MMP for $K_{\tilde{Y}} + F^B$ over Y (The notation F^B is in [3, 1.5] and $F^B = F^c$ in our case). Then we obtain $G\mathbb{Q}$-factorial dlt pair $f : (Y', B') \to (Y, B)$ over (Y, B). Since $K_{Y'} + B'$ is f-nef and (Y, B) is lc, we obtain that $K_{Y'} + B' = f^*(K_Y + B) \sim 0$. By [3, 2.4], B' has at most two connected components.

Definition 2.6 (for the local version, see [3, 4.12])**.** Let (Y, B) and $(\tilde{Y}, F)$ be as in 2.5. We define

$$\mu = \mu(Y, B) := \min\{\dim W \mid W \in CLC(\tilde{Y}, F)\}.$$

Note that in case B' is connected, then $0 \leq \mu \leq n - 1$ and in case B' has two connected components, then $\mu = n - 1$.

Case 1 (B' is connected)

There exist a G-isomorphism $\Gamma(Y, K_Y + B) \simeq \Gamma(Y', K_{Y'} + B')$ and an exact sequence:

$$0 = \Gamma(Y', K_{Y'}) \to \Gamma(Y', K_{Y'} + B') \to \Gamma(B', (K_{Y'} + B')|_{B'}) = \mathbb{C},$$

where the last term is isomorphic to $\Gamma(B', K_{B'})$, as $K_{Y'} + B'$ is a Cartier divisor. Therefore, we have only to check the action of G on $\Gamma(B', K_{B'})$.

Proposition 2.7 (for the local case, see [3, 4.11])**.** *If there exists a non-zero admissible section in* $\Gamma(B', m_0K_{B'})$*, then* G *acts on* $\Gamma(B', m_0K_{B'})$ *trivially.*

Proof. The proof is the same as that of [3, 4.11]. We have only to note that $B' = E = E^c$ in our case. □

Proposition 2.8 (for the local case, see [3, 4.14]). *Assume that* $\mu(Y,B) \le n-2$. *Then there exists a non-zero admissible section* $s \in \Gamma(B', m_0K_{B'})$ *with* $m_0 \in D_\mu$. *In particular,* s *is* G*-invariant. Thus,* $\mathrm{Ind}((Y,B)/G) \in I_\mu$.

Proof. The proof is the same as that of [3, 4.14]. Again $B' = E = E^c$. ☐

Proposition 2.9. *Assume that* B' *is connected and* $\mu(Y,B) = n-1$. *Then* $\mathrm{Ind}(Y,B)/G \in I'_{n-1}$.

Proof. In this case, B' is irreducible, therefore (Y', B') is plt. Then, by Adjunction [6, 17.6], B' is klt and $K_{B'} \sim 0$. Now apply 2.4. ☐

Case 2 (B' has two connected components).

Note that B' is the disjoint union of two irredicible components, therefore (Y', B') is plt (see [3, 2.4]). Run G-equivariant log MMP for $K + B' - \epsilon B'$, then we obtain a G-equivariant contraction $p : Y'' \to Z$ of an extremal face for $K + B'' - \epsilon B''$ to a lower dimensional variety Z, where $B'' = B''_1 \amalg B''_2$ is the divisor on Y'' corresponding to B'. Here $\dim Z = n-1$, because $h^{n-1}(Y', \mathcal{O}_{Y'}) = h^1(Y', K_{Y'}) \ne 0$. We also obtain that B''_i's are generic sections of p. Since (Y'', B'') is plt and $K_{Y''} + B'' \sim 0$, each B''_i has canonical singularities and $K_{B''_i} \sim 0$ by [6, 17.6]. Then the birational image Z has $K_Z \sim 0$, and therefore it has canonical singularities. Since the group $G = \langle g \rangle$ acts on B'', the subgroup $H := \langle g^2 \rangle$ acts on each B''_i $(i = 1, 2)$. Consider the exact sequence:

$$0 = \Gamma(Y'', K_{Y''} + B''_2) \to \Gamma(Y'', K_{Y''} + B'') \xrightarrow{\alpha} \Gamma(B''_1, K_{B''_1}),$$

where α is an H-equivariant isomorphism. On the other hand, the homomorphism $\Gamma(B''_1, K_{B''_1}) \to \Gamma(Z, K_Z)$ induced from $p|_{B''_1}$ is also an H-equivariant isomorphism. Hence, for two representations $\rho : G \to GL(\Gamma(Z, K_Z))$ and $\rho' : G \to GL(\Gamma(Y'', K_{Y''} + B''))$, we obtain the equality $|\rho(H)| = |\rho'(H)|$. Note that, for any representation $\lambda : G \to \mathbb{C}^*$, $\lambda(H) = \lambda(G)$ if and only if $|\lambda(G)|$ is an odd number. If we denote $|\rho(G)|$ by r, then $r \in I'_{n-1}$, and either: (1) $|\rho'(G)| = r$ or (2) $|\rho'(G)| = 2r$ and r is odd or (3) $|\rho'(G)| = r/2$ and $r/2$ is odd. By defining $I''_k := I'_k \cup \{2r \mid r \in I'_k \text{ odd}\} \cup \{r/2 \mid r \in I'_k, r/2 \text{ odd}\}$, we obtain:

Proposition 2.10. *Assume* B' *has two connected components. Then* $\mathrm{Ind}((Y,B)/G) \in I''_{n-1}$.

By 2.8, 2.9 and 2.10, we obtain Theorem 1.2. In particular, for the 3-dimensional case G-equivariant log MMP, the Abundance Theorem and (F'_j), (F_l) $(j = 2, l \le 1)$ hold. Here note that $I_0 = \{1, 2\}$, $I_1 =$

$\{1,2,3,4,6\}$ and $I_2' = \{r \in \mathbb{N} \mid \varphi(r) \leq 20, r \neq 60\}$ by [10] and [9]. By the list of the values of I_2' in [9, Table 1], we can check that $I_2'' = I_2'$. Therefore we obtain the necessary condition of the global index $\mathrm{Ind}(X, B_X)$ in Theorem 1.1.

The following shows that it is the sufficient condition of the global index:

Example 2.11. Let r be a positive integer that satisfies $\varphi(r) \leq 20$ and $r \neq 60$. Then by [8] and [9], there exists a $K3$-surface S with an action G of order r and $r = |\mathrm{Im}\rho|$. Let $Y = S \times \mathbb{P}^1$ and $B = S \times \{0\} + S \times \{\infty\}$. Let G act on Y by trivial action on $\mathbb{P}^1$ and the action above on S. Let (X, B_X) be the quotient of (Y, B) by G with $K_Y + B = \pi^*(K_X + B_X)$. Then (X, B_X) is a log Calabi-Yau 3-fold with global index r.

Remark 2.12. We can also prove Theorem 1.1 by using [4] instead of [3]. Indeed, we used [3] only for propositions 2.7 and 2.8. For the 3-dimensional case, these propositions can be replaced by the discussion on the order of the action of G on $H^2(F^B, \mathcal{O}_{F^B})$ for type $(0,0)$ and $(0,1)$. Theorems [4, 4.5] and [4, 4.12] give the same results as in 2.8.

Remark 2.13. Osamu Fujino informed the arthor that the boundedness of the indices of log Calabi-Yau 3-folds also follows from [3, 4.17] and the proof of [3, 4.14]. By this proof we obtain the index in I_2 instead of I_2'.

Remark 2.14. If we assume (F_n'), then it is clear that n-dimensional klt log Calabi-Yau variety has the global index $r \in I_n'$ by Lemma 2.4. Therefore klt log Calabi-Yau surface has the global index r such that $\varphi(r) \leq 20$ and $r \neq 60$.

For a klt log Calabi-Yau 3-fold with $B_X = 0$, the global index satisfies the same condition as above [9, Corollary 5].

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